

# Analysis of an Extended $\pm J$ Ising Spin Glass Model by Using a Gauge Symmetry

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We investigate an extended  $\pm J$  Ising spin glass model by using a gauge symmetry. This model has  $\pm J_1$  interactions and  $\pm J_2$  interactions. We show that a gauge symmetry is usable to study this model. The exact internal energy, the rigorous upper bound of the specific heat and some rigorous relations for correlation functions and order parameters are shown by using the gauge symmetry. The results are rigorous, and do not depend on any lattice shape. A part of our results, e.g., the value of the exact internal energy should be useful for checking the computer programs for investigating this model. In addition, we find that the present solutions are general solutions which include the solutions on the so-called Nishimori line for the conventional  $\pm J$  Ising spin glass model and the solutions for the bond-diluted  $\pm J$  Ising spin glass model.

The theoretical studies of spin glasses have been widely done.<sup>1),2),3)</sup> For the universality classes of spin glasses at zero temperature on the square lattice, the  $\pm J$  Ising spin glass model indicates that the stiffness exponent  $\theta$  is zero for the symmetric distribution of randomness,<sup>1),2)</sup> and the Gaussian Ising spin glass model indicates  $\theta \approx -0.28$  for the symmetric distribution of randomness.<sup>1),2)</sup> The stiffness exponent  $\theta$  is investigated from the system-size dependence of domain wall energy.<sup>2)</sup> The present model, i.e., an extended  $\pm J$  Ising spin glass model has  $\pm J_1$  interactions and  $\pm J_2$  interactions. When  $J_2/J_1 = 2$ , the  $\theta$  is zero for the symmetric distribution of randomness.<sup>2)</sup> Therefore, when  $J_2/J_1 = 2$ , the model has the universality class of the conventional  $\pm J$  Ising spin glass model. On the other hand, when  $J_2/J_1$  is  $(\sqrt{5} + 1)/2$  (the golden mean)  $\approx 1.618$ , the model is called the irrational model, and  $\theta \approx -0.29$  for the symmetric distribution of randomness is concluded.<sup>2)</sup> Therefore, the universality class of the irrational model is associated with the universality class of the Gaussian Ising spin glass model rather than that of the conventional  $\pm J$  Ising spin glass model. It is pointed out that this difference between the universality classes depends on the difference whether the energy level in the histogram for the domain wall energies is quantized or continuous.<sup>2)</sup> This property of the present model for universality class is remarkable. The studies for this model are seen in Refs.2), 4). We investigate this model by using a gauge symmetry.

Analysis by using gauge symmetries makes rigorous arguments possible. However, the applicable models are not so many. Here, the applicable models are models that physical quantities and/or relations are obtained by using gauge symmetries. As the applicable models, there are the conventional  $\pm J$  Ising spin glass model,<sup>5),6)</sup> the bond-diluted  $\pm J$  Ising spin glass model,<sup>7)</sup> the Gaussian Ising spin glass model,<sup>6)</sup> the Potts gauge glass model<sup>8)</sup> and the XY gauge glass model<sup>6),9)</sup> for example. We show that a gauge symmetry is usable to study the present model. In the applications of many other methods, a lattice shape is supposed in advance, and the results are calculated on the lattice. On the other hand, in this method using a gauge

symmetry, any lattice shape is not supposed in advance.

The Hamiltonian for the Edwards-Anderson Ising spin glass model,  $\mathcal{H}$ , is given by<sup>10)</sup>

$$\mathcal{H} = - \sum_{\langle i,j \rangle} J_{i,j} S_i S_j, \quad (1)$$

where  $\langle i, j \rangle$  denotes nearest-neighbor pairs,  $S_i$  is the state of the spin at a site  $i$ , and  $S_i = \pm 1$ .  $|J_{i,j}|$  is the strength of the exchange interaction between the spins at sites  $i$  and  $j$ . The value of  $J_{i,j}$  is given by a distribution  $P(J_{i,j})$  which is<sup>2)</sup>

$$P(J_{i,j}) = p \delta(J_{i,j} - J) + q \delta(J_{i,j} + J) + r \delta(J_{i,j} - aJ) + u \delta(J_{i,j} + aJ), \quad (2)$$

where  $J > 0$ , and  $\delta$  is the Dirac delta function.  $p$ ,  $q$ ,  $r$  and  $u$  are probabilities.  $p$  is the probability that the interaction has the strength of  $J$  and the interaction is ferromagnetic.  $q$  is the probability that the interaction has the strength of  $J$  and the interaction is antiferromagnetic.  $r$  is the probability that, if  $a > 0$ , the interaction has the strength of  $aJ$  and the interaction is ferromagnetic.  $r$  is the probability that, if  $a < 0$ , the interaction has the strength of  $|a|J$  and the interaction is antiferromagnetic.  $u$  is the probability that, if  $a > 0$ , the interaction has the strength of  $aJ$  and the interaction is antiferromagnetic.  $u$  is the probability that, if  $a < 0$ , the interaction has the strength of  $|a|J$  and the interaction is ferromagnetic.  $r + u$  is the probability that, if  $a = 0$ , the interaction is diluted. For  $p$ ,  $q$ ,  $r$  and  $u$ , we have

$$p + q + r + u = 1. \quad (3)$$

$a$  is a real number. When  $a = 1$  or  $-1$ , the model is the conventional  $\pm J$  Ising spin glass model. When  $a = 0$ , the model is the bond-diluted  $\pm J$  Ising spin glass model. When  $a = (\sqrt{5} + 1)/2$  or  $(\sqrt{5} - 1)/2$ , the model is the irrational model mentioned in Ref. 2).

To calculate thermodynamic quantities, a gauge transformation is used and is performed by<sup>5), 3), 6), 7), 11)</sup>

$$J_{i,j} \rightarrow J_{i,j} \sigma_i \sigma_j, \quad S_i \rightarrow S_i \sigma_i, \quad (4)$$

where  $\sigma_i$  is a variable at a site  $i$ , and takes 1 or  $-1$  arbitrarily. For the conventional Ising spin glass model, this gauge transformation has no effect on thermodynamic quantities.<sup>11)</sup> For the present model, also this gauge transformation has no effect on thermodynamic quantities. By using the gauge transformation, the  $\mathcal{H}$  part becomes  $\mathcal{H} \rightarrow \mathcal{H}$ .

By using Eq. (2), the distribution  $P(J_{ij})$  is rewritten as

$$P(J_{i,j}) = A e^{\beta_P^{(2)} J_{i,j}^2 + \beta_P J_{i,j}}, \quad (5)$$

$$A = \frac{1}{e^{\beta_P^{(2)} J^2 + \beta_P J} + e^{\beta_P^{(2)} J^2 - \beta_P J} + e^{\beta_P^{(2)} a^2 J^2 + \beta_P a J} + e^{\beta_P^{(2)} a^2 J^2 - \beta_P a J}}, \quad (6)$$

$$A e^{\beta_P^{(2)} J^2 + \beta_P J} = p, \quad (7)$$

$$A e^{\beta_P^{(2)} J^2 - \beta_P J} = q, \quad (8)$$

$$A e^{\beta_P^{(2)} a^2 J^2 + \beta_P a J} = r, \quad (9)$$

$$A e^{\beta_P^{(2)} a^2 J^2 - \beta_P a J} = u. \quad (10)$$

$\beta_P^{(2)}$ ,  $\beta_P$  and  $a$  are respectively

$$\beta_P^{(2)} = \frac{1}{2(1-a^2)J^2} \ln\left(\frac{pq}{ru}\right), \quad (11)$$

$$\beta_P = \frac{1}{2J} \ln\left(\frac{p}{q}\right), \quad (12)$$

$$a = \frac{\ln(r/u)}{\ln(p/q)}. \quad (13)$$

By using the gauge transformation, the distribution  $P(J_{i,j})$  part becomes

$$\begin{aligned} \prod_{\langle i,j \rangle} P(J_{i,j}) &= A^{N_B} e^{\beta_P^{(2)} \sum_{\langle i,j \rangle} J_{i,j}^2 + \beta_P \sum_{\langle i,j \rangle} J_{i,j}} \\ &\rightarrow \frac{A^{N_B}}{2^N} \sum_{\{\sigma_i\}} e^{\beta_P^{(2)} \sum_{\langle i,j \rangle} J_{i,j}^2 + \beta_P \sum_{\langle i,j \rangle} J_{i,j} \sigma_i \sigma_j}, \end{aligned} \quad (14)$$

where  $N$  is the number of sites, and  $N_B$  is the number of nearest-neighbor pairs in the whole system.

By using Eqs. (1) - (14), when the value of  $\beta_P$  is consistent with the value of the inverse temperature  $\beta$ , the exact internal energy,  $[\langle \mathcal{H} \rangle_{T_P}]_R$ , is obtained as

$$\begin{aligned} &[\langle \mathcal{H} \rangle_{T_P}]_R \\ &= \sum_{\langle i,j \rangle} \sum_{\{J_{l,m}\}} \prod_{\langle l,m \rangle} P(J_{l,m}) \frac{\sum_{\{S_l\}} (-J_{i,j} S_i, S_j) e^{\beta_P \sum_{\langle l,m \rangle} J_{l,m} S_l S_m}}{\sum_{\{S_l\}} e^{\beta_P \sum_{\langle l,m \rangle} J_{l,m} S_l S_m}} \\ &= \frac{A^{N_B}}{2^N} \sum_{\langle i,j \rangle} \sum_{\{J_{l,m}\}} \sum_{\{S_l\}} (-J_{i,j} S_i, S_j) e^{\beta_P^{(2)} \sum_{\langle l,m \rangle} J_{l,m}^2 + \beta_P \sum_{\langle l,m \rangle} J_{l,m} S_l S_m} \\ &= -J N_B \left[ p - q + \frac{\ln(r/u)}{\ln(p/q)} (r - u) \right], \end{aligned} \quad (15)$$

where  $\langle \rangle_{T_P}$  is the thermal average at the temperature  $T = T_P$ ,  $[\ ]_R$  is the random configuration average,  $\beta_P = 1/k_B T_P$ , and  $k_B$  is the Boltzmann constant. The value of the exact internal energy should be useful for checking the computer programs for investigating this model for example.

The specific heat  $C$  is given by

$$C = k_B \beta^2 ([\langle \mathcal{H}^2 \rangle_T]_R - [\langle \mathcal{H} \rangle_T^2]_R), \quad (16)$$

where  $\langle \rangle_T$  is the thermal average at the temperature  $T$ . By performing a similar calculation with the calculation in Eq. (15),  $[\langle \mathcal{H}^2 \rangle_{T_P}]_R$  is obtained as

$$\begin{aligned}
& [\langle \mathcal{H}^2 \rangle_{T_P}]_R \\
&= \sum_{\{J_{l,m}\}} \prod_{\langle l,m \rangle} P(J_{l,m}) \frac{\sum_{\{S_l\}} (\sum_{\langle i,j \rangle} J_{i,j} S_i S_j)^2 e^{\beta_P \sum_{\langle l,m \rangle} J_{l,m} S_l S_m}}{\sum_{\{S_l\}} e^{\beta_P \sum_{\langle l,m \rangle} J_{l,m} S_l S_m}} \\
&= N_B J^2 \left\{ p + q + \left[ \frac{\ln(r/u)}{\ln(p/q)} \right]^2 (r + u) \right\} \\
&\quad + N_B (N_B - 1) J^2 \left[ p - q + \frac{\ln(r/u)}{\ln(p/q)} (r - u) \right]^2. \tag{17}
\end{aligned}$$

By applying the Cauchy-Schwarz inequality, we obtain

$$[\langle \mathcal{H} \rangle_{T_P}^2]_R \geq [\langle \mathcal{H} \rangle_{T_P}]_R^2 = N_B^2 J^2 \left[ p - q + \frac{\ln(r/u)}{\ln(p/q)} (r - u) \right]^2. \tag{18}$$

By using Eqs. (16), (17) and (18), we obtain the rigorous upper bound of the specific heat at  $T = T_P$  as

$$\begin{aligned}
C &\leq k_B N_B \left( \ln \sqrt{\frac{p}{q}} \right)^2 [p + q - (p - q)^2] + k_B N_B \left( \ln \sqrt{\frac{r}{u}} \right)^2 [r + u - (r - u)^2] \\
&\quad - 2k_B N_B \left( \ln \sqrt{\frac{p}{q}} \right) \left( \ln \sqrt{\frac{r}{u}} \right) (p - q)(r - u). \tag{19}
\end{aligned}$$

Eq. (19) shows that the specific heat has no singularity when  $T = T_P$ .

It is straightforward to apply the same arguments as in the conventional  $\pm J$  Ising spin glass model case to derive identities and inequalities for correlation functions and order parameters. The results are rigorous and are obtained as

$$[\langle S_i S_j \rangle_T^n]_R = [\langle S_i S_j \rangle_{T_P} \langle S_i S_j \rangle_T^n]_R \quad (n = 1, 3, 5, \dots), \tag{20}$$

$$[P(m)]_R = [P(q)]_R \quad (T = T_P), \tag{21}$$

$$|[\langle S_i S_j \rangle_T]_R| \leq |[\langle S_i S_j \rangle_{T_P}]_R|, \tag{22}$$

$$[\text{sgn} \langle S_i S_j \rangle_T]_R \leq [\text{sgn} \langle S_i S_j \rangle_{T_P}]_R. \tag{23}$$

We omit the description of deriving Eqs. (20) - (23), since these applications are straightforward. See Ref. 3). When  $T = T_P$ , Eq. (20) with  $n = 1$  shows that the ferromagnetic correlation function on the left-hand side is equal to the spin glass correlation function on the right-hand side, and the limit  $|i - j| \rightarrow \infty$  yields  $m = q$ , where  $m$  and  $q$  are the magnetization and the spin glass order parameter respectively. The distribution function of the magnetization,  $[P(m)]_R$ , is

$$[P(m)]_R = \sum_{\{J_{i,j}\}} \prod_{\langle i,j \rangle} P(J_{i,j}) \frac{\sum_{\{S_i\}} \delta(Nm - \sum_l S_l) e^{\beta \sum_{\langle i,j \rangle} J_{i,j} S_i S_j}}{\sum_{\{S_i\}} e^{\beta \sum_{\langle i,j \rangle} J_{i,j} S_i S_j}}, \tag{24}$$

and the distribution function of the spin glass order parameter,  $[P(q)]_R$ , is

$$\begin{aligned}
& [P(q)]_R \\
&= \sum_{\{J_{i,j}\}} \prod_{\langle i,j \rangle} P(J_{i,j}) \\
& \times \frac{\sum_{\{S_i^{(1)}\}} \sum_{\{S_i^{(2)}\}} \delta(Nq - \sum_l S_l^{(1)} S_l^{(2)}) e^{\beta \sum_{\langle i,j \rangle} J_{i,j} (S_i^{(1)} S_j^{(1)} + S_i^{(2)} S_j^{(2)})}}{\sum_{\{S_i^{(1)}\}} \sum_{\{S_i^{(2)}\}} e^{\beta \sum_{\langle i,j \rangle} J_{i,j} (S_i^{(1)} S_j^{(1)} + S_i^{(2)} S_j^{(2)})}}, \quad (25)
\end{aligned}$$

where the spin  $S_i^{(k)}$  is the spin of the  $k$ -th replica at a site  $i$ . Eq. (21) shows that the distribution of the spin glass order parameter is consistent with the distribution of the magnetization when  $T = T_P$ . Eq. (22) shows that the phase boundary between the ferromagnetic and non-ferromagnetic phases below the multicritical point should be either vertical or reentrant in the phase diagram. Eq. (23) shows the spin pair becomes mutually parallel ignoring the magnitude when  $T = T_P$ .

The present results described above are given on condition that Eqs. (3), (12) and (13) are satisfied. The most part of the present results are not the results on the so-called Nishimori line, although a part of the present results are equivalent to the results on the Nishimori line. The present results are on a three-dimensional solid in a four-dimensional space consists of  $p$ ,  $q$ ,  $r$  and  $u$ . If  $p$  and  $q$  are fixed,  $T_P/J$  is also fixed. In addition, even if  $T_P/J$  is fixed, the results are affected by the value of  $a$ . The value of  $a$  depends on the values of  $r$  and  $u$  even if  $p$  and  $q$  are fixed. If we set  $p = x/2$ ,  $q = (1-x)/2$ ,  $r = x/2$ ,  $u = (1-x)/2$  and  $1 \geq x \geq 1/2$ , the results are results for  $a = 1$  and  $2\beta_P J = \ln[x/(1-x)]$ , and are equivalent to the results for the conventional  $\pm J$  Ising spin glass model, where  $x$  is the probability that the interaction is ferromagnetic, and  $1-x$  is the probability that the interaction is antiferromagnetic. In addition, if we set  $p = x/2$ ,  $q = (1-x)/2$ ,  $r = (1-x)/2$ ,  $u = x/2$  and  $1 \geq x \geq 1/2$ , the results are results for  $a = -1$  and  $2\beta_P J = \ln[x/(1-x)]$ , and are also equivalent to the results for the conventional  $\pm J$  Ising spin glass model. These results for the conventional  $\pm J$  Ising spin glass model are results on the Nishimori line, and are equivalent to the results in Refs 5), 6). If we set  $r = y/2$ ,  $u = y/2$  and  $1 > y > 0$ , the results are results for  $a = 0$  and  $2\beta_P J = \ln(p/q)$ , and are equivalent to the results for the bond-diluted  $\pm J$  Ising spin glass model, where  $y$  is the probability that the interaction is diluted. The results for the bond-diluted  $\pm J$  Ising spin glass model are equivalent to the results in Ref 7). Therefore, the present solutions are general solutions which include the solutions<sup>5), 6)</sup> for the conventional  $\pm J$  Ising spin glass model and the solutions<sup>7)</sup> for the bond-diluted  $\pm J$  Ising spin glass model.

An extended  $\pm J$  Ising spin glass model were investigated by using a gauge symmetry. This model has  $\pm J_1$  interactions and  $\pm J_2$  interactions. We showed that a gauge symmetry is usable to study this model. The exact internal energy, the rigorous upper bound of the specific heat and some rigorous relations for correlation functions and order parameters were shown by using the gauge symmetry. The results are rigorous, and do not depend on any lattice shape. A part of our results, e.g.,

the value of the exact internal energy should be useful for checking the computer programs for investigating this model. In addition, we found that the present solutions are general solutions which include the solutions<sup>5),6)</sup> on the Nishimori line for the conventional  $\pm J$  Ising spin glass model and the solutions<sup>7)</sup> for the bond-diluted  $\pm J$  Ising spin glass model.

For this article, it was pointed out that, in Ref. 6), it is already mentioned that this distribution (Eq. (2) in this article) is applicable for the local gauge transformation: this distribution is exactly the same form as Eq. (8) in Ref. 6) ( $P(J_{ij}) = f(J_{ij}) \exp(\beta_p J_{ij})$  and  $f(-J_{ij}) = f(J_{ij})$ ) which are mentioned as an applicable distribution of the local gauge transformation. We think this suggestion is correct. On the other hand, in Ref. 6), the gauge transformation is not explicitly applied to this distribution (this model), and, in Ref. 6), the exact internal energy and so forth for this model are not explicitly derived. Probably, this article is the first article for study of explicit application of the gauge transformation to this distribution (this model).

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